Additionnal antenna exercises

Exercise A1

We want to characterize a transmitting antenna operating at 100 MHz. When powered by a generator supplying 100 mW of power, a 10 cm long Hertz dipole placed 30 m away and optimally oriented receives a voltage of 10 mV in open circuit.

Find the maximum directivity of the transmitting antenna.

Note: We assume that the transmitting antenna is perfectly matched to the generator and has negligible loss resistance.

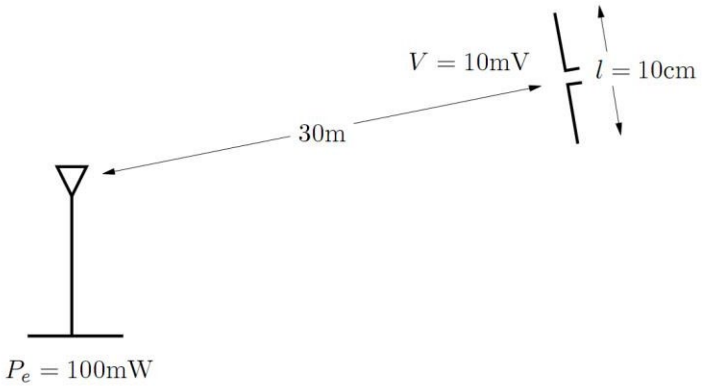


Figure 3. Described scene.

Solution :

According to the note, losses due to mismatch and Joule effect are negligible. Therefore:

Source Power ≡ Radiated Power

Here, Friis' formula for ideal antennas cannot be used directly, as the receiving antenna is open circuited.

The power density p of the incident plane wave is related to the electric field $\vec{E}$ by

$$p = \frac{|\vec{E}|^2}{Z_C} \tag{1}$$

where $Z_C = \sqrt{\mu_0/\varepsilon_0}$ is the characteristic impedance of air. Since the dipole is optimally oriented, in the far field the electric field lines of the incident wave are parallel to the dipole. Assuming that the dipole is small, its induced voltage is the product of the dipole length l and the tangential component of the field ($\vec{E}_t \equiv \vec{E}$).

$$U = |\vec{E}|l \rightarrow |\vec{E}| = \frac{U}{l} \tag{2}$$

On the other hand, power density appears in the directivity formula:

$$D = \frac{p}{p_{iso}} = \frac{4\pi r^2}{P_e}\frac{|E|^2}{Z_C} \tag{3}$$

where P_e is the transmitted power, D is the maximum directivity of the transmitting antenna, and r is the distance between the two antennas.

From equations (1), (2), and (3), we obtain the following for directivity

$$D = \frac{4\pi r^2 U^2}{Z_c l^2 P_e}$$

With r = 30 m, Pe = 0.1 W, l = 0.1 m, U = 0.1 V, and Zc = 120 Ω, the directivity of the transmitting antenna is

$$D = 3.$$

Exercise A2

We consider an antenna having a gain of 28.5 dBi and an equivalent noise temperature (antenna and receiver combined) of 100 K. How much will the G/T of this antenna increase if the losses in the cable linking the antenna to the receiver decrease by a factor of 0.1 dB? We consider that we are at room temperature, which corresponds at 20°C= 293K.

Solution:

The G/T of the system for an equivalent noise temperature of 100K is given by

$$\left(\frac{G}{T}\right)_{dB} = G_{dB} - T_{dB/K} = 28.5dB - 20dB/K = 8.5dB/K\ .$$

We obtain the equivalent noise temperature of a passive two port device (attenuator, cable) using (see course)

$$T_e = (L-1)T_{amb} \ .$$

Thus, a able with losses of L= 0.1 dB represents an equivalent noise temperature of

$$T_{loss,cable} = (L-1)T_{amb} = \left(10^{\frac{0.1}{10}} - 1\right) 293K = 6.82\,\mathrm{K}\,.$$

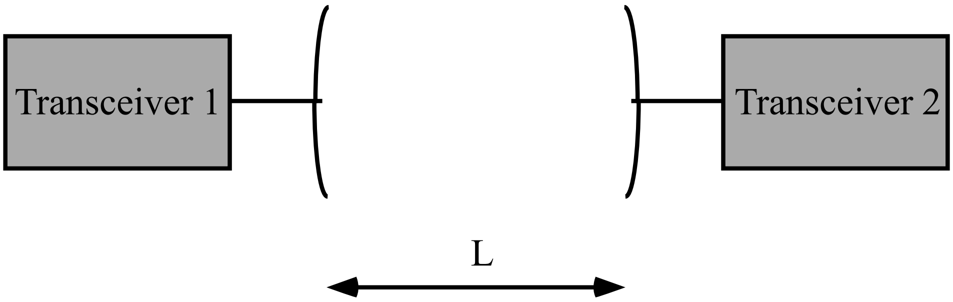
Thus, if we decrease the losses by 0.1dB the G/T becomes

$$\left(\frac{G}{T}\right)_{dB} = G_{dB} - (T - T_{loss})_{dB} = 28.5 - 10\log(100-6.82) = 28.5 - 19.7 = 8.8dB/K \ .$$

The G/T will increase by 0.3 dB/K.

Exercise A3

Consider the free space communication system shown in the following figure:



If antenna 1 has right-hand circular polarization (polarization vector given by $\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\\ 0\end{pmatrix}$ what will be the depolarization factor if antenna 2 has linear polarization?

And what becomes of this factor if antenna 2 has elliptical polarization given by

$$\boldsymbol{e_2} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ 1+j\\ 0\end{pmatrix}.$$

Solution:

$$\chi_{pol} = \left|\boldsymbol{e_1}\cdot\boldsymbol{e_2}^*\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ j\\ 0\end{pmatrix}\frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix}x\\ y\\ 0\end{pmatrix}\right|^2 = \frac{1}{2\left(x^2+y^2\right)}\left|x+jy\right|^2 = \frac{\left(x^2+y^2\right)}{2\left(x^2+y^2\right)} = \frac{1}{2}$$

$$\chi_{pol} = 0.5 = -3dB$$

and

$$\chi_{pol} = \left|\boldsymbol{e_1}\cdot\boldsymbol{e_2}^*\right|^2 = \left|\frac{1}{\sqrt{6}}\begin{pmatrix}1\\ j\\ 0\end{pmatrix}\begin{pmatrix}1\\ 1-j\\ 0\end{pmatrix}\right|^2 = \frac{1}{6}\left|1+j+1\right|^2 = \frac{5}{6} = -0.79dB$$

Exercise A4

Consider wireless communication in an urban environment (path loss exponent 3) with a data rate of 10 Mbits/s at a frequency of 5 GHz. The transmitter has a power of 2 W and an antenna with an effective area of 0.0028. The distance to the receiver is 500m, and the receiver has an antenna with a gain of 10. The effective temperature of the receiver (antenna + cable + amplifier) is 300K. What is the ratio of energy per bit to the noise power density of the receiver?

Solution:

Frequency = 5 GHz => λ= 0.06 m.

Therefore:

$$P(r)_{ic,th} = S_t\left(r_{\max},\theta_{inc},\varphi_{inc}\right)G_r\left(\theta_{inc},\varphi_{inc}\right)\frac{\lambda^2}{4\pi}\left(1-\left|\Gamma^*\right|^2\right)\chi_{pol} = \frac{EIRP}{4\pi^2}\frac{\lambda^2}{4\pi r_{\max}^{\ \ 2}}\chi_{pol}G_{r,real}\left(\theta,\varphi\right)$$

$$g = \frac{4\pi}{\lambda^2}A_e = 10 = 10\,dB \;\Rightarrow\; EIRP = 20W = 13dBW$$

The pathloss is given by

$$P_L = \left(\frac{4\pi D}{\lambda}\right)^3 = 1.14810^{15} = 150.6dB$$

The G/T if the receiver is given by

$$G_{dB} - T_{dB/K} = 10 - 10\log 300 = -14.8 dB/K$$

The signal-to-noise ratio at the receiver's input is therefore

$$\begin{aligned}\gamma|_{dBHz} &= EIRP|_{dBW} - \alpha_0|_{dB} + 228.6 + G_R|_{dB} - 10\cdot\log_{10} T_{op} \\ &= 13 - 150.6 + 228.6 + 10 - 24.8 = 76.2\, dBHz\end{aligned}$$

and the ratio of energy per bit to spectral noise density is given by

$$\eta = \frac{E_b}{N_0} = \frac{1}{R_b}\gamma$$

$$\eta_{dB} = \gamma_{dBHz} - 10\log_{10} R_b = 76.2 - 70 = 6.2 dB$$

Exercise A5

Two Hertzian dipoles are placed at the origin (0,0,0) of the coordinate system. One dipole is fed by a current $I_1 = e^{j0}$ and is oriented along the z-axis. The other is oriented along the x-axis and is fed by a current $I_2 = e^{j\pi/2}$. Both dipoles have the same length $l_1 = l_2 = l$.

1. Find the power radiation pattern in the yz-plane.
2. Determine the polarization of the electric field in the directions $\theta = 0°$, $45°$ and $90°$ within this plane.

Solution:

The first dipole is placed vertically along the z-axis, so its electric field can be written as:

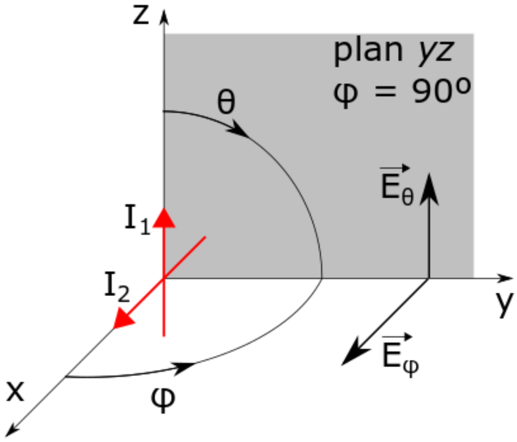
$$\vec{E}_1 = \frac{jZ_c}{2\lambda} I_1 l_1 \frac{e^{-jkr}}{r} \sin\theta\, \vec{e}_\theta$$

For the second dipole, which is oriented along the x-axis, the field expression is:

$$\vec{E}_2 = \frac{jZ_c}{2\lambda} I_2 l_2 \frac{e^{-jkr}}{r} \sin\phi\, \vec{e}_\phi$$

because of its orientation. We are only interested in the yz-plane, where:

$$\phi = \frac{\pi}{2}, \qquad \sin\phi = 1$$



Therefore, by adding the two fields ($\vec{E}_1$ and $\vec{E}_2$), we obtain:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{jZ_c}{2\lambda}\left(I_1 l_1 \sin\theta\, \vec{e}_\theta + I_2 l_2 \vec{e}_\phi\right)\frac{e^{-jkr}}{r}$$

Substituting $I_1 = e^{j0}$, $I_2 = e^{j\pi/2}$, and $l_1 = l_2 = l$:

$$\vec{E} = \frac{jZ_c}{2\lambda} l\left(\sin\theta\, \vec{e}_\theta + j\vec{e}_\phi\right)\frac{e^{-jkr}}{r} = E_0\left(\sin\theta\, \vec{e}_\theta + j\vec{e}_\phi\right)$$

The radiation pattern is then given by the power density, normalized with respect to its maximum value (p is proportional to à $\left(|E_\theta|^2 + |E_\phi|^2\right)$). Thus, the normalized directivity pattern is:

$$D_p = \left|\frac{\vec{E}(\theta,\phi)}{\vec{E}_{max}}\right|^2$$

where :

$$\left|\vec{E}(\theta,\phi)\right|^2 = \vec{E}(\theta,\phi)\cdot\vec{E}^*(\theta,\phi)$$
$$= |E_0|^2\left(\sin\theta\, \vec{e}_\theta + j\vec{e}_\phi\right)\cdot\left(\sin\theta\, \vec{e}_\theta - j\vec{e}_\phi\right)$$
$$= |E_0|^2(1 + \sin^2\theta)$$

Regarding the polarization, it is given by the field vector:

$$\vec{p} = \frac{\vec{E}}{|\vec{E}|} = \vec{p}_r + \mathrm{j}\vec{p}_i$$

We know from electromagnetic theory that:

$$\vec{p}_r \times \vec{p}_i = 0 \quad \Leftrightarrow \quad \text{linear polarization,}$$
$$\vec{p}_r \cdot \vec{p}_i = 0 \text{ et } |\vec{p}_r| = |\vec{p}_i| \quad \Leftrightarrow \quad \text{circular polarization,}$$
$$\text{otherwise} \quad : \quad \text{elliptical polarization.}$$

In our case:

$$\vec{p} = \frac{j}{\sqrt{1+\sin^2\theta}}\vec{e}_\phi + \frac{\sin\theta}{\sqrt{1+\sin^2\theta}}\vec{e}_\theta$$

Evaluating for each direction:

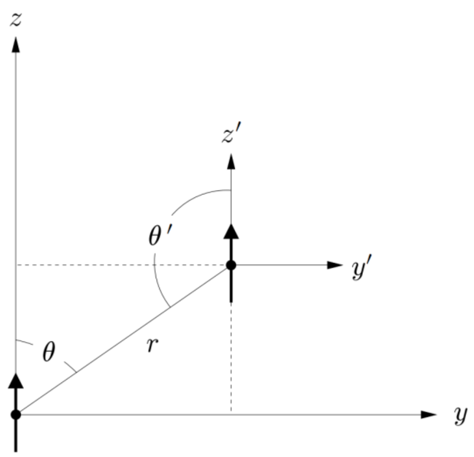
$$\theta = 0 \quad : \quad \vec{p}_i = \vec{e}_\phi \quad \text{and} \quad \vec{p}_r = 0 \quad \Rightarrow \text{linear polarization,}$$

$$\theta = \frac{\pi}{2} \quad : \quad \vec{p}_i = \frac{1}{\sqrt{2}}\vec{e}_\phi \quad \text{and} \quad \vec{p}_r = \frac{1}{\sqrt{2}}\vec{e}_\theta \quad \Rightarrow \text{circular polarization,}$$

$$\theta = \frac{\pi}{4} \quad : \quad \vec{p}_i = \sqrt{\frac{2}{3}}\vec{e}_\phi \quad \text{and} \quad \vec{p}_r = \frac{1}{\sqrt{3}}\vec{e}_\theta \quad \Rightarrow \text{elliptical polarization.}$$

Exercise A6

A transmission is established between two ideal Hertzian dipoles. One of the dipoles is fixed at the origin and oriented along the z-axis. The other dipole is also oriented along z, and it is located somewhere (away from the origin) in the plane defined by à (r,θ,φ = 90°) r≥0, 0°≤ θ ≤180° (that is, in the yz half-plane), and it is assumed to be in the far field of the first dipole (see figure below).



Find the dependence of the ratio of received power to transmitted power as a function of the angle θ.

Solution:

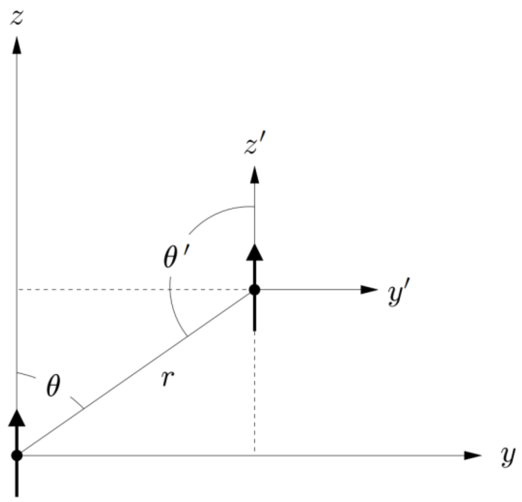


Figure 1. Given setup.

The ratio of received to transmitted power is calculated using Friis' transmission formula, taking into account the polarization of both antennas (by introducing the depolarization factor, FDP):

$$\frac{P_r}{P_e} = \text{FDP} \cdot D_1 D_2 \left(\frac{\lambda}{4\pi r}\right)^2$$

Both dipoles lie in the same plane (the yz-plane) and are oriented in the same direction ($\vec{e}_z$). Therefore, their polarizations are identical, and the depolarization factor is $\text{FDP} = 1$.

From the figure above, we observe that the angles θ' and θ are supplementary:

$$\theta' = \pi - \theta, \qquad 0 \leq \theta \leq \pi$$

The directivities of the two Hertzian dipoles are given by:

$$D_1(\theta) = \frac{3}{2}\sin^2\theta$$

$$D_2(\theta') = \frac{3}{2}\sin^2\theta' = \frac{3}{2}\sin^2\theta = D_1(\theta)$$

Therefore, the dependence of the power ratio P_r/P_e on the angle θ is given by the product of the two directivities:

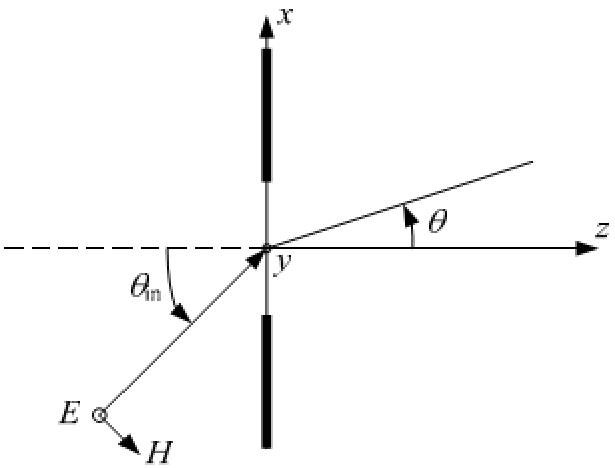
$$\frac{P_r}{P_e}(\theta) \propto D_1(\theta)D_2(\theta) \propto \sin^4\theta$$

since the other factors in Friis' formula are independent of θ.

Exercise A7

In a conducting plane located at z = 0 there is a slit (aperture) of length 2a along the x-axis, with a negligible width 2Δy along the y-axis. A plane wave impinges on the slit from the half-space z < 0 with an angle of incidence θ_{in} = 45° and with perpendicular polarization (the electric field is perpendicular to the plane of incidence xz).

1) Find the radiation pattern in the region z > 0 of the xz-plane (the plane of the page).
2) Sketch this pattern for the case 2a = 5 cm et f = 10 GHz.



Solution:

1) The electric field of the incident plane wave is given by:

$$\overrightarrow{E_{in}} = \hat{e}_y E_0 e^{-j\vec{\beta}\cdot\vec{r}}$$

Since it is a plane wave in free space, we have:

$$\vec{\beta} = \vec{k} = k(\hat{e}_x sin\theta_{in} + \hat{e}_z cos\theta_{in})$$

where $k = \frac{\omega}{c_0} = \omega\sqrt{\mu_0\varepsilon_0}$ and $\theta_{in} = \frac{\pi}{4} = 45°$

In the aperture plane (z = 0), we have:

$$\vec{r} = \hat{e}_x x + \hat{e}_y y \Rightarrow \vec{\beta}\cdot\vec{r} = kxsin\theta_{in} = \frac{kx}{\sqrt{2}}$$

Hence, the electric field in the aperture is:

$$\overrightarrow{E_{AP}} = \overrightarrow{E_{\imath n}}(z=0) = \hat{e}_y E_0 e^{-jkx/\sqrt{2}}$$

The magnetic current in the aperture is then:

$$\vec{M} = -2\hat{e}_z \times \overrightarrow{E_{AP}} = \hat{e}_x 2E_0 e^{-jkx/\sqrt{2}}$$

Thus, $\vec{M}$ has a constant magnitude across the slit but a varying phase. Moreover, $\vec{M}$ is longitudinal, i.e., directed along x (along the slit). This corresponds to the magnetic equivalent of a linear antenna.

The radiated electric field $\vec{E}$ produced by a magnetic current $\vec{M}$ is (see course notes):

$$\vec{E} = \frac{j}{4\lambda} \cdot \frac{e^{-jkr}}{r} \hat{e}_r \times \iint_{AP} \vec{M}\,(x', y') e^{j(k_x x' + k_y y')} dx' dy'$$

In the xz-plane (*ϕ=0*), we have:

$$\hat{e}_r = \hat{e}_x sin\theta + \hat{e}_z cos\theta$$

$$k_x = ksin\theta$$

$$k_y = 0$$

where :

$$\hat{e}_r \times \hat{e}_x = \hat{e}_y cos\theta$$

And

$$\vec{E} = \frac{jE_0}{2\lambda} \cdot \frac{e^{-jkr}}{r} \hat{e}_y cos\theta \int_{-a}^{a} e^{-\frac{jkx'}{\sqrt{2}}} e^{jk_x x'} dx' \cdot \int_{-\Delta y}^{\Delta y} dy'$$
$$= \hat{e}_y \frac{jE_0 \Delta y}{\lambda} \cdot \frac{e^{-jkr}}{r} sinc\left[ka\left(sin\theta - \frac{1}{\sqrt{2}}\right)\right] cos\theta$$

Like the incident wave, this field has only a y-component (≡ according to ϕ, since $\hat{e}_y = \hat{e}_\phi$ for *ϕ=0*).

Therefore, the radiation pattern $D_{E\phi}(\theta)$ is proportional to:

$$D_{E\phi}(\theta) = \frac{|E_\phi(\theta)|}{|E_\phi(\theta_{max})|} \propto \left| sinc\left[ka\left(sin\theta - \frac{1}{\sqrt{2}}\right)\right] cos\theta \right|$$

2) The minima of the radiation pattern correspond to:

- The zeros of the function: $cos\theta \Rightarrow \theta_{min} = \pm\frac{\pi}{2}$
- The zeros of the function: $sinc\left[ka\left(sin\theta - \frac{1}{\sqrt{2}}\right)\right] \Rightarrow ka\left(sin\theta_{min} - \frac{1}{\sqrt{2}}\right) = n\pi,\ \ n \in N^*$

Since $k = \frac{2\pi}{\lambda}$, the zeros are given by:

$$sin\theta_{min} = \frac{n\lambda}{2a} + \frac{1}{\sqrt{2}}$$

Thus, the larger 2a is compared to λ, he more zeros appear in the pattern — meaning more side lobes.

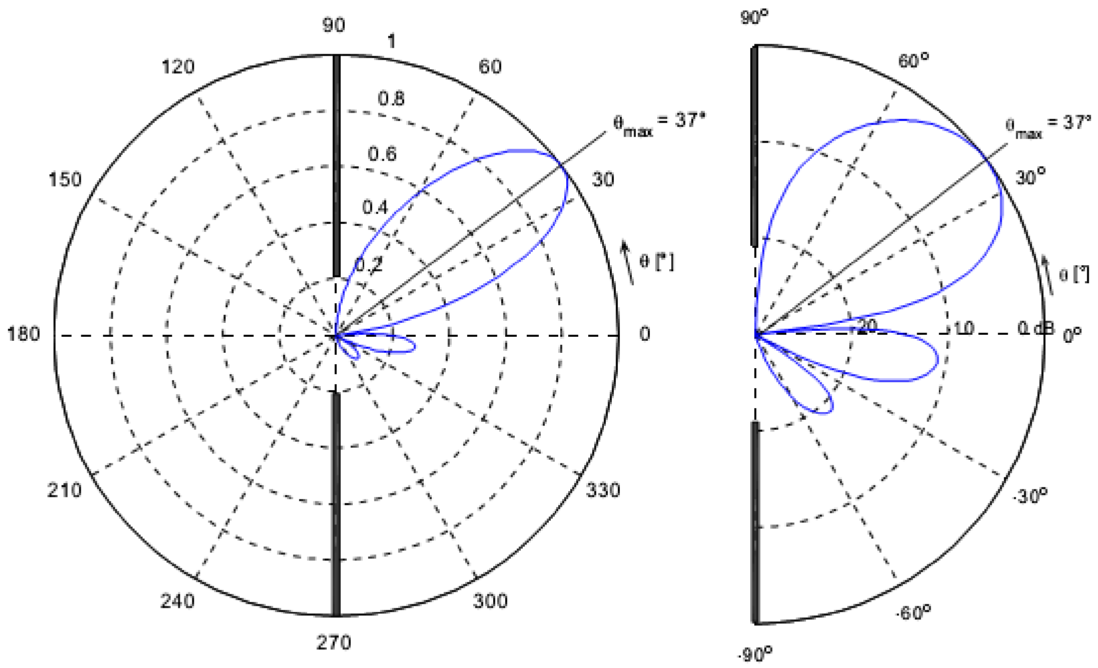
Conversely, if $\frac{2a}{\lambda} < \frac{1}{1+1/\sqrt{2}} \cong 0.59$, there are no zeros other than $\theta_{min} = \pm\frac{\pi}{2}$, and hence no side lobes.

The maxima of the radiation pattern are more difficult to determine analytically, since they require differentiating $D_{E\phi}(\theta)$, which is not straightforward.

If we neglect the cosθ dependence in $D_{E\phi}(\theta)$, the maximum occurs when the sinc function equals 1, i.e.:

$$sin\theta - \frac{1}{\sqrt{2}} = 0 \Rightarrow \theta_{max} = \frac{\pi}{4} = 45° = \theta_{in}$$

The radiation pattern is shown in Figure 1 for f=10 GHz and 2a=5 cm (given that λ = c/f = 3 cm and 2a/λ=1.67).



(a) (b)

Figure 1: Radiation pattern $D_{E\phi}(\theta)$ for f = 10 GHz and $2a$ = 5 cm. (a) Linear scale. (b) Logarithmic scale: 20 log $[D_{E\phi}(\theta))]$ [dB].

We observe that the maximum of the radiated field does not occur exactly at θ_{max} = 45°, but rather at θ_{max} = 37°, due to the element factor $cos\theta$.

Thus, this is a case where the vector nature of the field (its polarization) plays a significant role. It is also noted that for a wider slit, the sinc function would be narrower, and the influence of the $cos\theta$ factor on the location of the absolute maximum would be less significant.

Exercise A8

An antenna having a real input impedance $Z_{IN} = 100\Omega$ transmits an electric field, which in the far field looks as follows:

$$\vec{\mathbf{E}}(\theta,\varphi) = \begin{cases} \hat{\mathbf{e}}_{\boldsymbol{\theta}} C \cos\theta \exp(-jkr)/r & ; \quad \pi/4 < \theta < \pi/2 \\ 0 & ; \quad elsewhere \end{cases}$$

where C in a parameter which is independent of the coordinates.

a) Find the analytic expression for the total radiated power by this antenna.

b) What is the max. directivity and for which angle θ is it obtained ?

c) We excite this antenna with a current of 2 A. We measure at a distance of 10m and at an angle of $\theta = 60^\circ$ a power density of 1 W/m². What is the antenna efficiency?

d) We connect the antenna to a voltage generator of 10V having an internal impedance of 100 Ω. What is the power radiated by the antenna?

Solution :

a) The total radiated power is obtained by integrating the magnitude with the square of the Poynting vector over a unit sphere surrounding the antenna :

$$P_{tot} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta)\, r^2\, p(r,\theta,\phi), \qquad p(r,\theta,\phi) = \frac{1}{Z_c}\left|\vec{E}\right|^2$$

$$P_{tot} = \frac{1}{Z_c}\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left|\vec{E}\right|^2 r^2 \sin(\theta)\, d\theta$$

$$P_{tot} = \frac{2\pi C^2}{Z_c}\int_{\pi/4}^{\pi/2} \cos(\theta)^2 \sin(\theta)\, d\theta = \frac{\pi C^2}{3\sqrt{2}Z_c}$$

The quickest way to calculate this integral is to note that we have the cosine monomial multiplied by the derivative of the cosine.

b) The max directivity is obtained for $\theta = \pi/4$; we have the average radiated power density:

$$p_{iso} = \frac{P_{tot}}{4\pi r^2} = \frac{C^2}{12\sqrt{2}Z_c r^2}$$

From which the irectivity is geven by :

$$D_{max} = \frac{p(\theta = \pi/4)}{p_{iso}} = \frac{\left|\vec{E}(\theta = \pi/4)\right|^2}{Z_c p_{iso}} = \cos(\pi/4)^2\, 12\sqrt{2} = 6\sqrt{2}$$

c) Assuming the antenna is matched (since nothing is said about this) and taking into account the efficiency denoted η, the total radiated energy is:

$$\eta Z_{IN} I^2 = \frac{\pi C^2}{3\sqrt{2}Z_c}$$

The measurement performed allows the constant C to be isolated:

$$p\left(r = 10\text{ m}, \theta = \frac{\pi}{3}\right) = \frac{C^2 \cos\left(\frac{\pi}{3}\right)^2}{10^2 Z_c} = 1$$

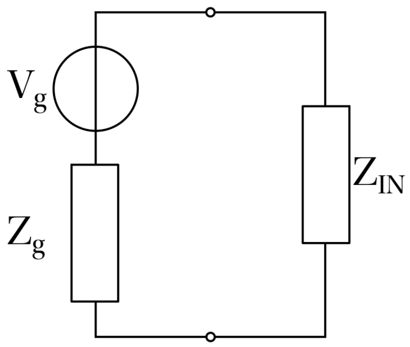
$$C^2 = 400 Z_c$$

Using this result in the previous equation, we have:

$$\eta = \frac{\pi}{3\sqrt{2}} = 0.74 = 74\% \, .$$

d) The circuit is modeled as two resistors in series, and the power dissipated in the radiation is determined using Kirchhoff's laws :

$$P_{tot} = \eta Z_{IN} \left(\frac{V_g}{Z_{IN} + Z_g}\right)^2 = 329\text{ mW}$$



Exercise A11

A 10 m diameter parabolic reflector transmitting antenna with an actual internal impedance of 10 ohms and an efficiency of 90% is fed by a 100 MHz frequency generator with a maximum available power of 10 watts and an internal impedance of 50 ohms.

a) What is the power radiated by the transmitting antenna?

b) This radiation is captured by placing a 30 cm long Hertz dipole receiving antenna in the direction of maximum radiation from the parabola. A 0.1 millivolt open-circuit signal is measured on this receiving antenna. What is the distance between the two antennas?

c) If the frequency is doubled, but the impedances of the generator and the transmitting antenna remain constant, what is the open-circuit voltage received by the dipole?

d) We return to the original frequency of 100 MHz, but we load the receiving dipole antenna with a load whose real impedance has a value of 20 ohms. What is the maximum power that can be dissipated in this load?

Solution:

Let d be the distance between the two antennas, l the length of the Hertzian dipole, U the measured voltage, and λ the wavelength.

a) The radiated power is, by definition:

$$P_{rad} = \eta(1 - |\Gamma|^2)P_{av}, \qquad \Gamma = \frac{Z_{in} - Z_g}{Z_{in} + Z_g}$$

The efficiency is given and the reflection coefficient is $|\Gamma| = \frac{2}{3}$. We find then $P_{rad} = 5$ W.

b) This is a very classicproblem. We proceed methodically:

- The effective aperture is given by $A_e = \frac{\pi D^2}{4} = 25\pi$, from which we get the directivity $D = 4\pi \frac{A_e}{\lambda^2} = \frac{(10\pi)^2}{\lambda^2}$.
- The measured voltage is related to the value of the electric field by $U = l|\vec{E}|$; We assume that there is no depolarization and that the field value is constant along the dipole.
- The power density at the Hertzian dipole can be calculated either via the electric field or by the directivity; by equating the two, the unknown can be isolated :

$$p = \frac{(10\pi)^2}{\lambda^2}\frac{P_{rad}}{4\pi d^2} = \frac{|\vec{E}|^2}{Z_c} \rightarrow d = \frac{5l}{U\lambda}\sqrt{Z_c P_{rad}\pi}$$

The distance between the antennas is 385 km. The wavelength is 3 m and can be easily calculated from the frequency.

c) If the frequency is doubled, the wavelength is halved. If no other parameters change, the previous formula shows that the electric field doubles. The measured voltage also doubles.

d) We use the formula from the course which allows us to find the radiation resistance of a Hertzian dipole as a function of its length and the working wavelength: $R_{rad} \approx 800\left(\frac{30\ cm}{3\ m}\right)^2 = 8\Omega$. The load is 20 Ω and the voltage applied to the equivalent circuit is 0.1 V; the power dissipated in the load is obtained using Kirchhoff's laws:

$$P_{charge} = 20\left(\frac{0.1}{20+8}\right)^2 = 2.5 * 10^{-4}\ \mathrm{W}$$

Exercise A12

Consider two Hertzian dipole antennas. The first is placed at point (0,0,d), oriented along the positive z-axis and fed by a current $I_1 = I$, , while the second is placed at point (0,0,2d), oriented along the positive x-axis and fed by a current of the same value. The plane is occupied by a perfect metallic surface, and the system operates at a frequency of 3 GHz.

a)) What is the value of the distance d that guarantees zero radiation in the positive z-axis direction $\theta = 0^{\circ}$? is the radiation in the opposite direction alos equal to zero (negative z axis $\theta = 180^{\circ}$)?

b) Can we have a circular polarization in the $\theta = 45^{\circ}$; $\varphi = 0^{\circ}$ (diagonal in plane *xz*) ? if yes, for which value of *d* ?

c) Can we have a circular polarization in the direction $\theta = 45^{\circ}$; $\varphi = 90^{\circ}$ (principal diagonal in plane *yz*) ? If yes, for which value of *d* ?

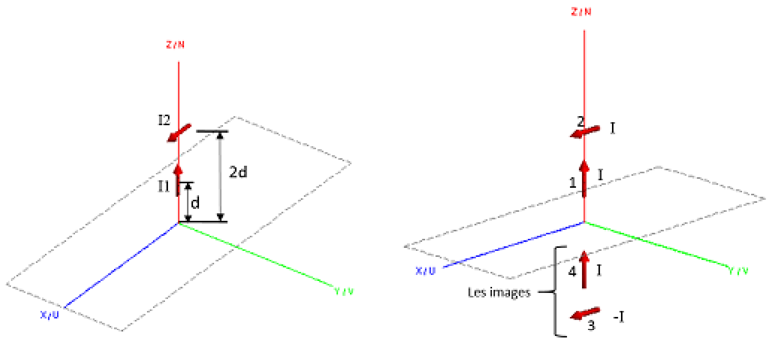


Figure 1: Les dipôles

Solution:

a) Image theory tells us that we can suppress the ground plane by introducing two image dipoles: The orientation is the opposite one for the horizontal dipole, and the same for the vertical dipole (see figure). The vertical dipoles do not contribute to the radiation for θ=0°. We can thus "guess" that the global radiation will be equal to zero in this direction when the fields off the two horizontal dipoles cancel each other, which happens when 4d=nλ. For physical reasons, there can be no radiation in the volume z<0, as this is below the ground plane.

b) We can quite often answer our physical intuation to answer questions, as was done above. If we consider the radiation in the xz plane (for instance along the drawn line), as the dipoles are located in this plane the electric field and the wave will also be located in this plane. Which means that the polarization can only be linear, and it is not possible to obtain a circular polarization

c) In this case the total field is given by the superposition of contribution of the two vertical dipoles, 1 and 4, and the two horizontal dipoles, 2 and 3. They are supported respectively by $\hat{\varphi}$ and $\hat{\theta}$, which are orthogonal to each other. In order to have circular polarization, we need two orthogonal fields with same amplitude and a phase shift of 90°. This can be created by the horizontal and vertical dipoles, respectively, in the following way:

$$E_\varphi = CA_1(\theta,\varphi)\frac{e^{-jkr}}{r}\left(e^{jk2d\cos(45°)} - e^{-j2d\cos(45°)}\right) = 2jCA_1(\theta,\varphi)\sin\left(k2\cos(45°)\right)$$

$$E_\theta = CA_2(\theta,\varphi)\frac{e^{-jkr}}{r}\left(e^{jk2d\cos(45°)} + e^{-j2d\cos(45°)}\right) = 2jCA_2(\theta,\varphi)\cos\left(k2\cos(45°)\right)$$

C is a constant which is the same for all dipoles. Moreover, $A_1(\theta,\varphi) = \sqrt{1-\sin^2\theta\cos^2\varphi}$ and $A_2(\theta,\varphi) = \sin\theta$ are the radiation patterns of a horizontal dipole located along the x axis and a vertical dipole along the z axis, respectively. We see that they are in quadrature. Thus, the only remaining condition to obtain circular polarization is that their amplitude is the same:

$$\left|2CA_2(\theta=45°,\varphi=90°)\cos(kd\cos(45°))\right| = \left|2CA_1(\theta=45°,\varphi=90°)\sin(kd\cos(45°))\right|$$

$$\Rightarrow \left|\frac{\sqrt{2}}{2}\cos(\alpha)\right| = \left|\sin(2\alpha)\right| \quad , \quad \alpha = kd\cos(45°)$$

We thus obtain

$$\sin\alpha = \pm\frac{\sqrt{2}}{4} \Rightarrow \alpha = \pm 0.3614 + n\pi$$

$$k = \frac{2\pi}{\lambda} = 62.83$$

$$d = \frac{\alpha}{k\cos(45°)} = \frac{\pm 0.3614 + n\pi}{44.42}$$

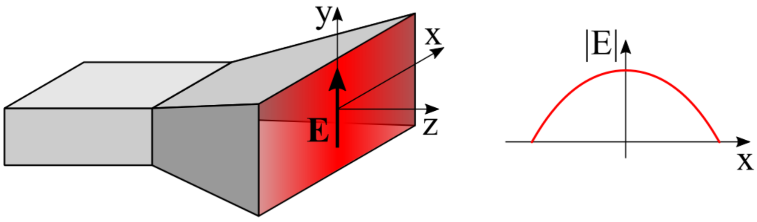
Keeping only positive values for d (which is a distance)

Exercise A13

A horn antenna with dimensions a and b is shown in the figure. The electric field distribution on the aperture of the horn can be (very roughly) approximated by:

$$\mathbf{E} = E_0 \cos\left(\frac{\pi x}{a}\right) \mathbf{e}_\mathrm{y}$$

a) Calculate the radiated electric field in the *yz* plane.
b) Calculate the angles that correspond to the minima of the radiation pattern in the *yz* plane.
c) If the aperture efficiency (A_{ef}/A) of this antenna is 81%, calculate the maximum antenna directivity in dB for $a = \lambda, b = \lambda/2$.



Solution

a) The exercise can be solved by finding the radiation of equivalent magnetic currents on the horn aperture. We replace the electric field with an equivalent magnetic current:

$$\mathbf{M_s} = -\mathbf{n} \times \mathbf{E} = -\mathbf{e}_\mathrm{z} \times \mathbf{E} = E_0 \cos\left(\frac{\pi x}{a}\right) \mathbf{e}_\mathrm{x}$$

from which we can obtain the radiated electric field:

$$\mathbf{E} = j\omega Z_c \mathbf{e}_\mathrm{r} \times \mathbf{A}_\mathrm{m}$$

The vector potential for magnetic sources $\mathbf{A}_\mathrm{m}$ (sometimes labeled **F**) is in the general case:

$$\mathbf{A}_\mathrm{m}(\mathbf{r}) = \frac{\epsilon}{4\pi} \frac{e^{-jkr}}{r} \int_V \mathbf{M}(\mathbf{r}') e^{jk\mathbf{e}_\mathrm{r}\cdot\mathbf{r}'} dv'$$

In this specific case, the magnetic currents exist only on the aperture surface:

$$\mathbf{A}_\mathrm{m} = \frac{\epsilon_0}{4\pi} \frac{e^{-jkr}}{r} \int_V \mathbf{M_s}(\mathbf{r}') e^{jk\mathbf{e}_\mathrm{r}\cdot\mathbf{r}'} dS' = \frac{\epsilon_0}{4\pi} \frac{e^{-jkr}}{r} \iint_S \mathbf{M_s}(\mathbf{r}') e^{jk\mathbf{e}_\mathrm{r}\cdot\mathbf{r}'} dS'$$

We have that:

$$k\mathbf{e}_\mathrm{r} \cdot \mathbf{r}' = \mathbf{k} \cdot \mathbf{r}' = k_x x' + k_y y' + k_z z'$$

$$k_x = k \sin\theta \cos\phi\,, \qquad k_y = k \sin\theta \sin\phi\,, \qquad k_z = k \cos\theta$$

In the horn aperture, the *z* coordinate is equal to 0, and in the *yz* plane, the ϕ angle is equal to 90°. Therefore:

$$\mathbf{k} \cdot \mathbf{r}' = k \sin\theta\, y'$$

We can separate the surface integral into a double integral, knowing that $dS' = dx'dy'$:

$$\mathbf{A}_{\mathrm{m}} = \frac{\epsilon_0}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{jk\sin\theta y'}dy' \int_{-\frac{a}{2}}^{\frac{a}{2}} E_0 \cos\left(\frac{\pi x}{a}\right)\mathbf{e}_{\mathrm{x}}\,dx'$$

$$\mathbf{A}_{\mathrm{m}} = -\frac{abE_0\epsilon_0}{2\pi^2}\frac{e^{-jkr}}{r}\operatorname{sinc}\left(\frac{kb}{2}\sin\theta\right)\mathbf{e}_{\phi}$$

since $\mathbf{e}_{\mathrm{x}} = -\mathbf{e}_{\phi}$ in the *yz* plane. From there, the radiated electric field is:

$$\mathbf{E} = j\omega Z_c \mathbf{e}_{\mathrm{r}} \times \mathbf{A}_{\mathrm{m}}$$

$$\mathrm{E}_{\theta} = \frac{jabE_0}{\lambda}\frac{e^{-jkr}}{r}\operatorname{sinc}\left(\frac{kb}{2}\sin\theta\right)$$

$$\mathbf{E} = \mathrm{E}_{\theta}\mathbf{e}_{\theta}, \qquad \theta > 0$$

$$\mathbf{E} = -\mathrm{E}_{\theta}\mathbf{e}_{\theta}, \qquad \theta < 0$$

b)

$$\theta_{min} = \operatorname{asin}\left(\frac{\lambda}{b}\right)$$

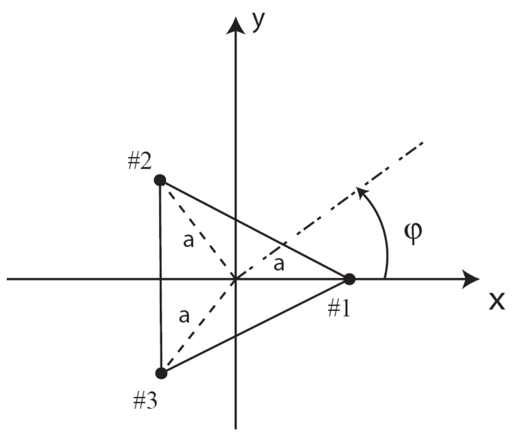
c)

$$D = \frac{4\pi}{\lambda^2}A_{ef} = \frac{4\pi}{\lambda^2}0.81ab = 1.62\pi = 5.09 = 7.07\text{ dB}$$

Exercise A14

Three elementary antennas are arranged in a triangular array, each occupying a vertex of an equilateral triangle. The antennas lie in the horizontal xy plane (θ=90°) and are positioned at points (x, y):P_1 (a,0) ; P_2 $\left(-\frac{a}{2}, a\frac{\sqrt{3}}{2}\right)$; P_3 $\left(-\frac{a}{2}, -a\frac{\sqrt{3}}{2}\right)$

Find the mathematical expression for the lattice factor (AF) in the xy plane as a function of the polar coordinate φ.



Numerical application :

Considerr a=10 cm.
Fibd the value of the Array factor in the direction φ=0 and φ=60 degrees,
For the 3 following cases:

	I_1	I_2	I_3	f (GHz)
Case #1	1	1	1	1
Case #2	1	1	1	2
Case #3	1	-1	-1	1

For the very motivated :
Draw the radiation pattern in the xy plane for the three cases seen above.

Solution:

The array factor in its general form is given by:

$$\mathrm{AF} = \sum_{n=0}^{N-1} I_n e^{jk\hat{e}_r \cdot \vec{d}_n}$$

where I_n is the current excitation of element n, $\hat{e}_r$ is the radial unitary vector in spherical coordinates towards the observation point and $\vec{d}_n$, is the vector form the global coordinate system to element n. Remember that :

$$\hat{e}_r = \cos\varphi \sin\theta \hat{x} + \sin\varphi \sin\theta \hat{y} + \cos\theta \hat{z}$$

In plane ($\theta = 90^\circ$) we have $\sin\theta = 1, \cos\theta = 0$. If we substitute this in he general formula and particularize for:

$$\vec{d}_1 = a\hat{x}, \quad \vec{d}_2 = -\frac{a}{2}\hat{x} + a\frac{\sqrt{3}}{2}\hat{y} \text{ and } \vec{d}_2 = -\frac{a}{2}\hat{x} - a\frac{\sqrt{3}}{2}\hat{y}$$

We simplify the general formula to obtain :

$$AF(\theta,\varphi)=I_1e^{jka\cos\varphi}+I_2e^{jka(-\frac{1}{2}\cos\varphi+\frac{\sqrt{3}}{2}\sin\varphi)}+I_3e^{jka(-\frac{1}{2}\cos\varphi-\frac{\sqrt{3}}{2}\sin\varphi)}$$

Case 1a: $\varphi=0$, $I_1=I_2=I_3=1$, f=1GHz:

$$k=\frac{2\pi f}{c}=20.944(rad/m),\quad AF(\theta=90^{\circ},\varphi=0^{\circ})=\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

Case 1b: $\varphi=60^{\circ}$, $I_1=I_2=I_3=1$, f=1GHz:

$$k=\frac{2\pi f}{c}=20.944(rad/m),\quad AF(\theta=90^{\circ},\varphi=60^{\circ})=\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

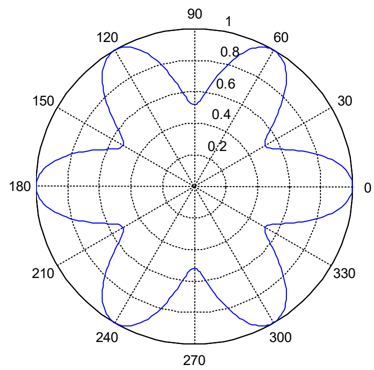


Figure 2: NAF for case #1

Case 2a: $\varphi=0$, $I_1=I_2=I_3=1$, f=2GHz:

$$k=\frac{2\pi f}{c}=41.8879(rad/m),\quad AF(\theta=90^{\circ},\varphi=0^{\circ})=-1.5-2.5981i$$

Case 2b: $\varphi=60^{\circ}$, $I_1=I_2=I_3=1$, f=2GHz:

$$\mathrm{k}=\frac{2\pi \mathrm{f}}{\mathrm{c}}=41.8879(rad/m), \quad AF(\theta=90^{\circ},\varphi=60^{\circ})=-1.5+2.5981i$$

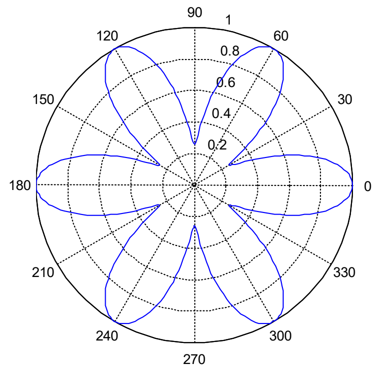


Figure 3: NAF for case #2

Case 3a: $\varphi=0^{\circ}$, I_1=1, I_2=I_3=-1, f=1GHz:

$$\mathrm{k}=\frac{2\pi \mathrm{f}}{\mathrm{c}}=20.944(rad/m), \quad AF(\theta=90^{\circ},\varphi=0^{\circ})=-1.5+2.5981i$$

Case 3b: $\varphi=60^{\circ}$, I_1=1, I_2=I_3=-1, f=1GHz:

$$\mathrm{k}=\frac{2\pi \mathrm{f}}{\mathrm{c}}=20.944(rad/m), \quad AF(\theta=90^{\circ},\varphi=60^{\circ})=0.5+0.866i$$

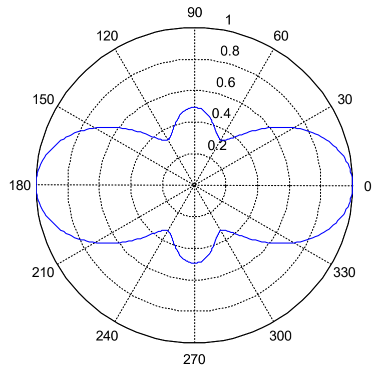


Figure 3: NAF for case #3